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$R(C_6, K_5) = 21$ and $R(C_7, K_5) = 25$

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The Ramsey number $R(C_n, K_m)$ is the smallest integer p such that any graph G on p vertices either contains a cycle C_n with length n or contains an independent set with order m. In this paper we prove that $R(C_n, K_5) = 4(n-1)+1$ $(n = 6, 7)$.

1. INTRODUCTION

We shall only consider graphs without multiple edges or loops.

The Ramsey number $R(C_n, K_m)$ is the smallest integer p such that any graph G on p vertices either contains a cycle C_n with length n or contains an independent set with order m.

In 1976, Shelp and Faudree in [9] stated the following problem.

PROBLEM 1.1 ([9]). Find the range of integers n and m such that $R(C_n, K_m) = (n-1)(m-1)+1$. In particular, does the equality hold for $n \geq m$?

For this problem, the following results are known:

$R(C_4, K_4) = 10$ (see [2])
$R(C_4, K_5) = 14$ (see [3])
$R(C_5, K_4) = 13$, $R(C_5, K_5) = 17$ (see [5, 6])
$R(C_n, K_3) = 2n-1$ $(n > 3)$ (see [4, 7]).

In [10], we proved that $R(C_n, K_4) = 3(n-1)+1$ $(n \geq 4)$. In this paper, we will prove that $R(C_n, K_5) = 4(n-1)+1$ $(n = 6, 7)$.

The following notations will be used in this paper. If G is a graph, the vertex set (resp. edge set) of G is denoted by $V(G)$ (resp. $E(G)$). For $x \in V(G)$, $N(x) = \{v \in V(G) | xv \in E(G)\}$. If $V \subset V(G)$, then $N(V) = \bigcup_{x \in V} N(x)$.

A cycle with n vertices $x_1, x_2, \ldots, x_n$ will be denoted by

$$C_n = C_n(x_1, x_2, \ldots, x_n)$$

where the subscript i in x_i will be taken modulo the cycle length n.

For $n, m \geq 1$, a (C_n, K_m)-graph is a graph without cycles of length n or independent sets of order m, a (C_{n+1}, K_m)-graph G is called a

$$(C_{n+1}, K_m; C_n(x_1, x_2, \ldots, x_n), I_{m-1}(y_1, y_2, \ldots, y_{m-1}))\text{-graph}$$

if $C_n(x_1, x_2, \ldots, x_n)$ is a subgraph of G, and $I_{m-1}(y_1, y_2, \ldots, y_{m-1})$ is an independent set of order $m-1$ in G, where

$$I_{m-1}(y_1, y_2, \ldots, y_{m-1}) = \{y_1, y_2, \ldots, y_{m-1}\} \subset V(G) - \{x_1, x_2, \ldots, x_n\}.$$

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2. LEMMAS

In this section, we assume that G is a

$$(C_{n+1}, K_m; C_n(1, 2, \ldots, n), I_{m-1}(x_1, x_2, \ldots, x_{m-1}))\text{-graph}.$$

For convenience, we denote $I_{m-1}(x_1, x_2, \ldots, x_{m-1})$ by I_{m-1}, and assume that $n \geq m$.

LEMMA 2.1. (1) *$N(i) \cap I_{m-1} \neq \emptyset$ for $i \in \{1, 2, \ldots, n\}$;*
(2) *$|N(x) \cap \{i, i+1\}| \leq 1$ for $x \in I_{m-1}$.*

PROOF. It is clear that (1) is true. (2) is same as Lemma 1.3(a) of [10]. □

LEMMA 2.2 (CF. [10], LEMMA 1.3(C)). *Let $x \in I_{m-1}$. If $\{i, j\} \subset N(x)(i \neq j, i \neq j \pm 1(mod\ n))$, then*

$$|N(y) \cap \{i+1, j+2\}| \leq 1, |N(y) \cap \{j-1, i-2\}| \leq 1$$

for $y \in I_{m-1} - \{x\}$.

LEMMA 2.3. *Let $x \in I_{m-1}$. If $\{i, j\} \subset N(x)(i \neq j, i \neq j \pm 1(\text{mod } n))$, then:*
(1) *$i-1 \notin N(j-1), i+1 \notin N(j+1)$;*
(2) *there is a $z_1 \in N(i-1) \cap (I_{m-1} - \{x\})$, and a $z_2 \in N(j-1) \cap (I_{m-1} - \{x\})$ such that $z_1 \neq z_2$;*
(3) *there is a $z_1 \in N(i+1) \cap (I_{m-1} - \{x\})$, and a $z_2 \in N(j+1) \cap (I_{m-1} - \{x\})$ such that $z_1 \neq z_2$.*

PROOF. (1) see [10, Lemma 1.3(b)].
(2) If $N(i-1) \cap I_{m-1} \neq N(j-1) \cap I_{m-1}$, the conclusion of (2) is clear by Lemma 2.1(2). Now, we assume that $N(i-1) \cap I_{m-1} = N(j-1) \cap I_{m-1}$, then we have the following two cases.

Case a. $|N(i-1) \cap I_{m-1}| \geq 2$.

By Lemma 2.1(2), since $i \in N(x)$, we obtain $N(i-1) \cap I_{m-1} = N(i-1) \cap (I_{m-1} - \{x\})$. Let $\{z_1, z_2\} \subset N(i-1) \cap I_{m-1}$ with $z_1 \neq z_2$, then z_1 and z_2 satisfy the conclusion of (2).

Case b. $|N(i-1) \cap I_{m-1}| = 1$.

By (1), we have that $\{i-1, j-1\} \cup \{I_{m-1} - N(i-1)\}$ is an independent set of order m in G, a contradiction. Therefore $|N(i-1) \cap I_{m-1}| \neq 1$.
By Cases a and b, (2) is true. Similarly, (3) is true. □

LEMMA 2.4. *Let $x \in I_{m-1}$. If $n \geq 2m-3$ and $|N(x) \cap \{1, 2, \ldots, n\}| = k$, then $k \leq m-3$.*

PROOF. For convenience, we assume that $N(x) \cap \{1, 2, \ldots, n\} = \{i_1, i_2, \ldots, i_k\}$. By Lemma 2.3, we know that $\{i_1+1, i_2+1, \ldots, i_k+1\}$ is an independent set. Now we have

$$|N(\{i_1+1, i_2+1, \ldots, i_k+1\}) \cap I_{m-1}| \geq k,$$

otherwise

$$(I_{m-1} - N(\{i_1+1, i_2+1, \ldots, i_k+1\})) \cup \{i_1+1, i_2+1, \ldots, i_k+1\}$$

is an independent set with order $\geq m$, a contradiction.

Since $n \geq 2m - 3$, we may assume that $i_k + 2 \neq i_1 (\mathrm{mod}\ n)$. Now, by Lemma 2.2, we have

$$N(\{i_1 + 1, i_2 + 1, \ldots, i_k + 1\}) \cap I_{m-1} \cap N(i_k + 2) = \emptyset.$$

Since $N(x) \cap \{1, 2, \ldots, n\} = \{i_1, i_2, \ldots i_k\}$, we have

$$m - 1 \geq |(N(\{i_1 + 1, i_2 + 1, \ldots, i_k + 1\}) \cap I_{m-1}) \cup N(i_k + 2) \cup \{x\}| \geq k + 2,$$

i.e., $k \leq m - 3$. □

The following theorem can be found in [2].

THEOREM 2.5 ([2]). *Let F_1 and F_2 be two graphs with no isolated vertices. Let c be the number of vertices in a largest connected component of F_1, and let χ be the chromatic number of F_2. Then the following lower bound holds:*

$$R(F_1, F_2) \geq (c - 1)(\chi - 1) + 1.$$

THEOREM 2.6 ([10]). $R(C_n, K_4) = 3(n - 1) + 1$ $(n \geq 4)$.

3. $R(C_6, K_5) = 21$

In this section we assume that G is a graph with order 21. In the following, we will prove that G either contains a cycle of length 6 or contains an independent set of order 5. For convenience, we suppose to the contrary that G is a (C_6, K_5)-graph. Now, by $R(C_5, K_5) = 17$, we may assume that $C_5(1, 2, 3, 4, 5)$ is a cycle of G. Since $|V(G) - \{1, 2, 3, 4, 5\}| = 16$ and by Theorem 2.6, we may assume that $I_4(x_1, x_2, x_3, x_4)$ is an independent set of G and $I_4(x_1, x_2, x_3, x_4) \subset V(G) - \{1, 2, 3, 4, 5\}$, i.e., G is a $(C_6, K_5; C(1, 2, \ldots, 5), I_4(x_1, \ldots, x_4))$-graph.

It is clear $d(v) \geq 5$ for $v \in V(G)$.

LEMMA 3.1. *If* $\{1, 4\} \subset N(x_1)$, $2 \in N(x_2)$, $5 \in N(x_3)$, *then:*

(1) $\{1, 3, 5\} \cap N(x_2) = \emptyset$;
(2) $\{1, 2, 3, 4\} \cap N(x_3) = \emptyset$;
(3) $\{2, 4, 5\} \cap N(x_4) = \emptyset$ *and* $3 \in N(x_4)$;
(4) $\{2, 3, 5\} \cap N(x_1) = \emptyset$.

PROOF. (1) $5 \notin N(x_2)$, otherwise $C_6(x_2, 2, 1, x_1, 4, 5)$ is a cycle of G, a contradiction. By $2 \in N(x_2)$ and Lemma 2.1, we have $\{1, 3\} \cap N(x_2) = \emptyset$. Thus we obtain $\{1, 3, 5\} \cap N(x_2) = \emptyset$.

(2) $2 \notin N(x_3)$, otherwise $C_6(x_3, 2, 1, x_1, 4, 5))$ is a cycle of G, a contradiction. By $\{1, 4\} \subset N(x_1)$, $5 \in N(x_3)$ and Lemma 2.2, we have $3 \notin N(x_3)$. Thus we obtain $\{1, 2, 3, 4\} \cap N(x_3) = \emptyset$.

(3) By (1) and (2), we know that $3 \notin N(x_2, x_3)$; by $4 \in N(x_1)$, we have $3 \notin N(x_1)$. Thus we obtain $3 \in N(x_4)$. Using the same methodology as (1), we have $\{2, 4, 5\} \cap N(x_4) = \emptyset$.

(4) By $\{1, 4\} \subset N(x_1)$ and Lemma 2.1, we have $\{2, 3, 5\} \cap N(x_1) = \emptyset$. □

LEMMA 3.2. *If* $1 \in N(x_1)$, $2 \in N(x_2)$, $5 \in N(x_3)$, *then* $4 \notin N(x_1)$.

PROOF. Suppose that $4 \in N(x_1)$. By Lemma 3.1, we have $3 \in N(x_4)$.

Since $d(x_1) \geq 5$ and $N(x_1) \cap (I_4 \cup \{1, 2, 3, 4, 5\}) = \{1, 4\}$ by Lemma 3.1.

Thus there are two vertices in $V(G) - I_4 \cup \{1, 2, 3, 4, 5\}$, say z_1 and z_2, such that $z_1, z_2 \in N(x_1)$.

Claim 1. $\{z_1, z_2\} \cap N(x_2, x_4) = \emptyset$.

$z_1 \notin N(x_2)$, otherwise $C_6(z_1, x_1, 4, 3, 2, x_2)$ is a cycle of G, a contradiction;
$z_1 \notin N(x_4)$, otherwise $C_6(z_1, x_1, 1, 2, 3, x_4)$ is a cycle of G, a contradiction.

Thus we obtain $z_1 \notin N(x_2, x_4)$. Similarly, $z_2 \notin N(x_2, x_4)$. Thus $\{z_1, z_2\} \cap N(x_2, x_4) = \emptyset$.

Claim 2. $1 \in N(x_4), 4 \in N(x_2)$.

Suppose that $1 \notin N(x_4)$.

$z_1 \notin N(1)$, otherwise $C_6(z_1, x_1, 4, 3, 2, 1)$ is a cycle of G, a contradiction. By Claim 1, we have $z_1 \notin N(x_2, x_4)$, by Lemma 3.1, we have $1 \notin N(x_2, x_3)$. Thus $\{1, x_2, x_3, x_4\}$ and $\{z_1, 1, x_2, x_4\}$ are independent sets of G. This implies that $z_1 \in N(x_3)$, otherwise $\{z_1, 1, x_2, x_3, x_4\}$ is an independent set of G, a contradiction. Similarly, we have $z_2 \in N(x_3)$ and $z_2 \notin N(1)$. Now, we have $z_1 \notin N(z_2)$, otherwise $C_6(z_1, x_1, 1, 5, x_3, z_2)$ is a cycle of G, a contradiction. Hence $\{z_1, z_2, 1, x_2, x_4\}$ is an independent set of G, a contradiction. Thus $1 \in N(x_4)$. Similarly, $4 \in N(x_2)$

By Claim 2, we find that $C_6(x_2, 2, 1, x_4, 3, 4)$ is a cycle of G, a contradiction. Hence $4 \notin N(x_1)$. □

LEMMA 3.3. *If* $1 \in N(x_1)$, *then* $N(x_1) \cap \{2, 3, 4, 5\} = \emptyset$.

PROOF. It is clear that $\{2, 5\} \cap N(x_1) = \emptyset$. If $4 \in N(x_1)$, by Lemma 2.3, we may assume that $2 \in N(x_2), 5 \in N(x_3)$. Now, by Lemma 3.2, we have $4 \notin N(x_1)$, a contradiction. Thus $4 \notin N(x_1)$. Similarly, $3 \notin N(x_3)$. Now, we have $N(x_1) \cap \{2, 3, 4, 5\} = \emptyset$. □

THEOREM 3.4. $R(C_6, K_5) = 21$.

PROOF. By Lemma 3.3, the number of edges joining I_4 and $\{1, 2, 3, 4, 5\}$ is ≤ 4, by Lemma 2.1, the number of edges joining $\{1, 2, 3, 4, 5\}$ and I_4 is ≥ 5, a contradiction. Thus G either contains a cycle of length 6 or an independent set of order 5, i.e., $R(C_6, K_5) \leq 21$. On the other hand, by Theorem 2.5, we have $R(C_6, K_5) \geq 21$. Thus $R(C_6, K_5) = 21$. □

4. $R(C_7, K_5) = 25$

In this section we assume that G is a graph with order 25. For convenience, we suppose that G is a (C_7, K_5)-graph. Now, by $R(C_6, K_5) = 21$, we may assume that $C_6(1, 2, 3, 4, 5, 6)$ is a cycle of G. Since $|V(G) - \{1, 2, 3, 4, 5, 6\}| = 19$ and Theorem 2.6, we may assume that $I_4(x_1, x_2, x_3, x_4)$ is an independent set of G and $I_4(x_1, x_2, x_3, x_4) \subset V(G) - \{1, 2, 3, 4, 5\}$, i.e., G is a

$$(C_7, K_5; C(1, 2, \ldots, 6), I_4(x_1, \ldots, x_4))\text{-graph.}$$

It is clear, $d(v) \geq 6$, otherwise $V(G) \setminus (I_4 \cup \{1, 2, \ldots, 6\})$ contains either C_7 or a 4-element independent set, a contradiction.

LEMMA 4.1. $|\{1, 2, 3, 4, 5, 6\} \cap N(x_i)| \leq 2$ *for* $i = 1, 2, 3, 4$.

PROOF. Suppose to the contrary that there is a vertex in I_4, say x_1, such that $|\{1, 2, 3, 4, 5, 6\} \cap N(x_1)| \geq 3$. It is clear we may assume that $\{1, 3, 5\} \subset N(x_1)$. Furthermore, by Lemma 2.3, we can assume that $2 \in N(x_2), 6 \in N(x_3)$.

Claim 1. $4 \notin N(x_2, x_3), 4 \in N(x_4)$.

If $4 \in N(x_3)$, then $3 \notin N(5)$ by Lemma 2.3. Furthermore, we have $1 \notin N(3)$, otherwise $C_7(x_1, 1, 3, 4, x_3, 6, 5)$ is a cycle of G, a contradiction; $\{1, 3\} \cap N(x_2) = \emptyset$ since $\{1, 3\} \subset N(x_1)$; $5 \notin N(x_2)$ since $2 \in N(x_2)$ and $\{1, 3\} \subset N(x_1)$.

Thus $\{1, 3, 5, x_2, x_3\}$ is an independent set of G, a contradiction.

Therefore we obtain $4 \notin N(x_3)$. Similarly, $4 \notin N(x_2)$. Now, we have $4 \in N(x_4)$.

Claim 2. $G(\{1, 3, 5\}) = K_3$, $\{2, 4, 6\}$ is an independent set of G.

$1 \in N(5)$, otherwise by Lemma 2.2 $\{x_2, x_3, x_4, 1, 5\}$ is an independent set of G, a contradiction. Similarly $1 \in N(3)$, $3 \in N(5)$, i.e., $G(\{1, 3, 5\}) = K_3$.

$\{2, 4, 6\}$ is an independent set of G is trivial by Lemma 2.3.

Note that $N(x_1) \cap \{1, 2, \ldots, 6\} = \{1, 3, 5\}$ and $d(x_1) \geq 6$, Thus $(V(G)-(\{1, 2, 3, 4, 5, 6\} \cup I_4)) \cap N(x_1) \neq \emptyset$. Let t_1 be a vertex in $(V(G)-(\{1, 2, 3, 4, 5, 6\} \cup I_4)) \cap N(x_1)$, then we have $t_1 \notin N(x_3)$, otherwise $C_7(t_1, x_1, 1, 3, 5, 6, x_3)$ is a cycle of G, a contradiction. Similarly $t_1 \notin N(x_2, x_4)$. It is clear, $1 \notin N(x_2, x_3, x_4)$ by Lemmas 2.1 and 2.3, i.e., $\{x_2, x_3, x_4, 1\}$ is an independent set of G, thus we obtain $t_1 \in N(1)$. But, in this case, we have $C_7(t_1, x_1, 3, 4, 5, 6, 1)$ is a cycle of G, a contradiction.

Now, the lemma is true. □

LEMMA 4.2. *If* $\{1, 5\} \subset N(x_1)$, $2 \in N(x_2)$, $6 \in N(x_3)$, *then* $4 \notin N(x_3)$.

PROOF. Suppose to the contrary that $4 \in N(x_3)$. Now we have:

$2 \notin N(4)$, otherwise $C_7(x_1, 1, 2, 4, x_3, 6, 5)$ is a cycle of G, a contradiction;
$4 \notin N(6)$ by Lemma 2.3 and $\{1, 5\} \subset N(x_1)$.

Thus we obtain $\{2, 4, 6, x_1\}$ as an independent set of G. And by Lemma 4.1, we have $3 \notin N(x_1)$, $3 \notin N(x_3)$. Note that $3 \notin N(x_2)$ by $2 \in N(x_2)$. Hence, we have $3 \in N(x_4)$. By this we find that $2, 4 \notin N(x_4)$ and, by Lemma 2.2, $6 \notin N(x_4)$. Therefore $\{2, 4, 6, x_1, x_4\}$ is an independent set of G, a contradiction. Thus, we obtain the lemma. □

LEMMA 4.3. *If* $1 \in N(x_1)$, *then* $3, 5 \notin N(x_1)$.

PROOF. Suppose to the contrary that $5 \in N(x_1)$. Now, by Lemma 2.3, we may assume that $2 \in N(x_2)$ and $6 \in N(x_3)$. Then we have $3 \notin N(x_1)$ by Lemma 4.1; $3 \notin N(x_3)$ by Lemma 2.2. It is clear that $3 \notin N(x_2)$. Thus we obtain $3 \in N(x_4)$. Furthermore, we have $4 \notin N(x_3)$ by Lemma 4.2, $4 \notin N(x_1, x_4)$. Thus we obtain $4 \in N(x_2)$.

Claim 1. $1 \in N(5)$, $2 \in N(4)$; $\{1, 2, \ldots, 6\} \cap N(x_1) = \{1, 5\}$, $\{1, 2, \ldots, 6\} \cap N(x_2) = \{2, 4\}$, $\{1, 2, \ldots, 6\} \cap N(x_3) = \{6\}$ and $\{1, 2, \ldots, 6\} \cap N(x_4) = \{3\}$.

Since $6 \in N(x_3)$, we have $1, 5 \notin N(x_3)$. By Lemma 2.2 and $\{1, 5\} \subset N(x_1)$, we have $3 \notin N(x_3)$. By Lemma 4.2, we have $2, 4 \notin N(x_3)$. Thus $\{1, 2, \ldots, 6\} \cap N(x_3) = \{6\}$. Similarly, $\{1, 2, \ldots, 6\} \cap N(x_4) = \{3\}$.

Now, if $1 \notin N(5)$, we have $\{1, 5, x_2, x_3, x_4\}$ as an independent set of G, a contradiction. Thus $1 \in N(5)$. Similarly, $2 \in N(4)$.

It is clear that $\{1, 2, \ldots, 6\} \cap N(x_1) = \{1, 5\}$. Similarly, $\{1, 2, \ldots, 6\} \cap N(x_2) = \{2, 4\}$.

By Claim 1, $N(x_1) \cap \{1, 2, \ldots, 6\} = \{1, 3, 5\}$. Since $d(x_1) \geq 6$, thus $|(V(G) - (\{1, 2, 3, 4, 5, 6\} \cup I_4)) \cap N(x_1)| \geq 3$.

Now, we may assume $z_1, z_2 \in (V(G) - (\{1, 2, 3, 4, 5, 6\} \cup I_4)) \cap N(x_1)$. Thus:

$z_1 \notin N(1)$, otherwise $C_7(z_1, x_1, 5, 4, 3, 2, 1)$ is a cycle of G, a contradiction;
$z_1 \notin N(x_2)$, otherwise $C_7(z_1, x_1, 5, 4, 3, 2, x_2)$ is a cycle of G, a contradiction;
$z_1 \notin N(x_4)$, otherwise $C_7(z_1, x_1, 1, 5, 4, 3, x_4)$ is a cycle of G, a contradiction;
$z_1 \in N(x_3)$, otherwise $\{z_1, 1, x_2, x_4, x_3\}$ is an independent set of G, a contradiction.

Using this we obtain $z_1 \notin N(1, x_2, x_4)$ and $z_1 \in N(x_3)$. Similarly, $z_2 \notin N(1, x_2, x_4)$ and $z_2 \in N(x_3)$. If $z_1 \notin N(z_2)$, then $\{z_1, z_2, 1, x_2, x_4\}$ is an independent set of G, a contradiction. Thus $z_1 \in N(z_2)$, and then we have $C_7(z_1, x_1, 1, 5, 6, x_3, z_2)$ is a cycle of G, a contradiction.

Therefore $5 \notin N(x_1)$. Similarly, $3 \notin N(x_1)$. □

THEOREM 4.4. $R(C_7, K_5) = 25$.

PROOF. By Lemma 2.1, we know that there is a vertex in I_4, say x_1, such that $|\{1, 2, \ldots, 6\} \cap N(x_1)| \geq 2$. Now, by Lemmas 2.1 and 4.3, we have $\{1, 2, \ldots, 6\} \cap N(x_1) = \{1, 4\}$.

Using Lemmas 2.1 and 4.1, we may assume that $|\{2, 3, 5, 6\} \cap N(x_2)| \geq 2$, without loss of generality, $6 \in N(x_2)$. Now we have $\{1, 2, \ldots, 6\} \cap N(x_2) = \{3, 6\}$ by Lemma 4.3.

By Lemma 2.3 and $\{1, 4\} \subset N(x_1)$, we have $|N(6, 3) \cap I_4| \geq 2$. Now, we may assume that $x_3 \in N(6, 3)$. Thus by Lemma 4.3 we have $N(x_3) \cap \{1, 2, \ldots, 6\} \subset \{6, 3\}$.

By the above, we obtain $2, 5 \in N(x_4)$. And $1 \notin N(4)$ by Lemma 2.3. Thus $\{1, 4, x_2, x_3, x_4\}$ is an independent set of G, a contradiction.

Therefore we obtain $R(C_7, K_5) \leq 25$. On the other hand, by Theorem 2.5, we have $R(C_7, K_5) \geq 25$. Thus $R(C_7, K_5) = 25$. □

NOTE. In [1], we also proved that $R(C_n, K_5) = 4(n-1) + 1$ $(n \geq 5)$.

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